

## WEAK COMPACTNESS OF ALMOST LIMITED OPERATORS

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ABSTRACT. The paper is devoted to the relationship between almost limited operators and weakly compact operators. We show that if  $F$  is a  $\sigma$ -Dedekind complete Banach lattice then, every almost limited operator  $T : E \rightarrow F$  is weakly compact if and only if  $E$  is reflexive or the norm of  $F$  is order continuous. Also, we show that if  $E$  is a  $\sigma$ -Dedekind complete Banach lattice then the square of every positive almost limited operator  $T : E \rightarrow E$  is weakly compact if and only if the norm of  $E$  is order continuous.

## 1. INTRODUCTION

Throughout this paper  $X, Y$  will denote real Banach spaces, and  $E, F$  will denote real Banach lattices.  $B_X$  is the closed unit ball of  $X$  and  $B_E^+ := B_E \cap E^+$  is the positive part of  $B_E$ . We will use the term operator  $T : X \rightarrow Y$  between two Banach spaces to mean a bounded linear mapping. We refer to [1, 5] for unexplained terminology of the Banach lattice theory and positive operators.

Let us recall that a norm bounded set  $A$  in a Banach space  $X$  is called *limited*, if every weak\* null sequence  $(f_n)$  in  $X^*$  converges uniformly to zero on  $A$ , that is,  $\sup_{x \in A} |f_n(x)| \rightarrow 0$ . An operator  $T : X \rightarrow Y$  is said to be *limited* whenever  $T(B_X)$  is a limited set in  $Y$ , equivalently, whenever  $\|T^*(f_n)\| \rightarrow 0$  for every weak\* null sequence  $(f_n) \subset Y^*$ .

Recently, the authors of [2] considered the disjoint version of limited sets by introducing the class of almost limited sets in Banach lattices. From [2] a norm bounded subset  $A$  of a Banach lattice  $E$  is said to be *almost limited*, if every disjoint weak\* null sequence  $(f_n)$  in  $E^*$  converges uniformly to zero on  $A$ .

From [4], an operator  $T : X \rightarrow E$  is called *almost limited* if  $T(B_X)$  is an almost limited set in  $E$ , equivalently,  $\|T^*(f_n)\| \rightarrow 0$  for every disjoint weak\* null sequence  $(f_n) \subset E^*$ . Note that an almost limited operator need not be weakly compact. In fact, the identity operator of the Banach lattice  $\ell^\infty$  is almost limited but it is not weakly compact.

In this paper, we characterize pairs of Banach lattices  $E, F$  for which every almost limited operator  $T : E \rightarrow F$  is weakly compact. More precisely, we will prove that if  $F$  is a  $\sigma$ -Dedekind complete Banach lattice then, every almost limited operator  $T : E \rightarrow F$  is weakly compact if and only if  $E$  is reflexive or the norm of  $F$  is order continuous (Theorem 2.5). Next, we will prove that if  $E$  is a  $\sigma$ -Dedekind complete Banach lattice then the square of every positive almost limited operator  $T : E \rightarrow E$  is weakly compact if and only if the norm of  $E$  is order continuous (Theorem 2.9). As consequences, we will give some interesting results.

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## 2. MAIN RESULTS

Let us recall that a Banach lattice  $E$  is said to have the dual positive Schur property if  $\|f_n\| \rightarrow 0$  for every weak\* null sequence  $(f_n) \subset (E^*)^+$ , equivalently,  $\|f_n\| \rightarrow 0$  for every weak\* null sequence  $(f_n) \subset (E^*)^+$  consisting of pairwise disjoint terms (Proposition 2.3 of [8]). A Banach lattice  $E$  has the property (d) whenever  $|f_n| \wedge |f_m| = 0$  and  $f_n \xrightarrow{w^*} 0$  in  $E^*$  imply  $|f_n| \xrightarrow{w^*} 0$ . It should be noted, by Proposition 1.4 of [8], that every  $\sigma$ -Dedekind complete Banach lattice has the property (d) but the converse is not true in general. In fact, the Banach lattice  $\ell^\infty/c_0$  has the property (d) but it is not  $\sigma$ -Dedekind complete [8, Remark 1.5].

Our first result shows that we can restrict sequences appearing in the definition of almost limited operator  $T : X \rightarrow E$  to positive disjoint sequences if the Banach lattice  $E$  has the property (d).

**Proposition 2.1.** *An operator  $T : X \rightarrow E$  from a Banach space  $X$  into a Banach lattice  $E$  with the property (d), is almost limited if and only if  $\|T^*(f_n)\| \rightarrow 0$  for every weak\* null sequence  $(f_n)$  in  $E^*$  consisting of positive and pairwise disjoint elements.*

*Proof.* The “only if” part is trivial. For the “if” part, let  $(f_n) \subset E^*$  be a disjoint weak\* null sequence. As  $E$  has the property (d),  $|f_n| \xrightarrow{w^*} 0$ . Using the inequalities  $0 \leq f_n^+ \leq |f_n|$  and  $0 \leq f_n^- \leq |f_n|$ , we see that  $(f_n^+)$  and  $(f_n^-)$  are disjoint weak\* null sequences of  $E^*$ . So, from our hypothesis we see that  $\|T^*(f_n^+)\| \rightarrow 0$  and  $\|T^*(f_n^-)\| \rightarrow 0$ . This implies that  $\|T^*(f_n)\| \rightarrow 0$ , and hence  $T$  is almost limited.  $\square$

The next result follows immediately from Proposition 2.3 of [8] combined with Proposition 2.1.

**Corollary 2.2.** *A Banach lattice  $E$  with the property (d) has the dual positive Schur property if and only if the identity operator on  $E$  is almost limited.*

The following result shows that if a positive almost limited operator  $T : E \rightarrow F$  has its range in a Banach lattice with the property (d), then every positive operator  $S : E \rightarrow F$  that it dominates (i.e.,  $0 \leq S \leq T$ ) is also almost limited.

**Proposition 2.3.** *Let  $E$  and  $F$  be two Banach lattices such that  $F$  has the property (d). If a positive operator  $S : E \rightarrow F$  is dominated by an almost limited operator, then  $S$  itself is almost limited.*

*Proof.* Let  $S, T : E \rightarrow F$  be two operators such that  $0 \leq S \leq T$  and  $T$  is almost limited. Let  $(f_n)$  be a disjoint sequence in  $(F^*)^+$  such that  $f_n \xrightarrow{w^*} 0$ . As  $T$  is almost limited,  $\|T^*(f_n)\| \rightarrow 0$ . Using the inequalities  $0 \leq S^*(f_n) \leq T^*(f_n)$ , we see that  $\|S^*(f_n)\| \leq \|T^*(f_n)\|$  for all  $n$ , from which we get  $\|S^*(f_n)\| \rightarrow 0$ . Now, by Proposition 2.1  $S$  is well almost limited.  $\square$

The next remark will be useful in further considerations.

**Remark 2.4.** (1) *Consider the scheme of operators  $X \xrightarrow{R} Y \xrightarrow{S} F$ . It is easy to see that if  $S$  is an almost limited operator, then  $S \circ R$  is likewise almost limited.*

(2) *Consider the scheme of operators  $X \xrightarrow{R} E \xrightarrow{S} F$ .*

- (a) If  $R$  is an almost limited operator, then  $S \circ R$  is not necessarily almost limited. In fact, by a result in [6], there exists a non regular operator  $S : \ell^\infty \rightarrow c_0$ , which is certainly not compact. So by Proposition 4.3 of [4],  $S$  is not almost limited. If  $R : \ell^\infty \rightarrow \ell^\infty$  is the identity operator on  $\ell^\infty$  then  $R$  is almost limited but  $S \circ R = S$  is not almost limited.
- (b) However, if  $E$  has the dual positive Schur property (for example,  $E = \ell^\infty$ ) and  $F$  has the property (d), and  $S$  is positive, then  $T = S \circ R$  is an almost limited operator. In fact, according to Proposition 2.1, let  $(f_n) \subset F^*$  be a positive disjoint weak\* null sequence. Clearly  $0 \leq S^* f_n \xrightarrow{w^*} 0$  holds in  $E^*$ . Since  $E$  has the dual positive Schur property then  $\|S^* f_n\| \rightarrow 0$ , and hence  $\|T^* f_n\| = \|R^*(S^* f_n)\| \rightarrow 0$ , as desired.

Our next major result characterizes pairs of Banach lattices  $E, F$  for which every positive almost limited operator  $T : E \rightarrow F$  is weakly compact.

**Theorem 2.5.** *Let  $E$  and  $F$  be two Banach lattices such that  $F$  is  $\sigma$ -Dedekind complete. Then the following assertions are equivalent:*

- (1) Every almost limited operator  $T : E \rightarrow F$  is weakly compact.
- (2) Every positive almost limited operator  $T : E \rightarrow F$  is weakly compact.
- (3) One of the following statements is valid:
  - (a)  $E$  is reflexive.
  - (b) The norm of  $F$  is order continuous.

*Proof.* (1)  $\Rightarrow$  (2) Obvious.

(2)  $\Rightarrow$  (3) Assume by way of contradiction that  $E$  is not reflexive and the norm of  $F$  is not order continuous. We have to construct a positive almost limited operator  $T : E \rightarrow F$  which is not weakly compact.

Indeed, since the norm of  $F$  is not order continuous, then by Corollary 2.4.3 of [5] we may assume that  $\ell^\infty$  is a closed sublattice of  $F$ . As  $E$  is not reflexive then  $E^*$  is not reflexive, and hence the closed unit ball  $B_{E^*}$  of  $E^*$  is not weakly compact. So, from  $B_{E^*} \subset B_{E^*}^+ - B_{E^*}^+$ , we see that  $B_{E^*}^+$  is not weakly compact. Then, by the Eberlein-Šmulian theorem one can find a sequence  $(f_n)$  in  $B_{E^*}^+$  which does not have any weakly convergent subsequence. Consider the positive operator  $T : E \rightarrow \ell^\infty \subset F$  defined by

$$T(x) = (f_n(x))_{n=1}^\infty$$

for all  $x \in E$ . By Remark 2.4(2b)  $T$  is an almost limited operator. But  $T$  is not weakly compact. In fact, if  $T$  were weakly compact then  $T^* : (\ell^\infty)^* \rightarrow E^*$  would be weakly compact. Note that  $T^*((\lambda_n)_{n=1}^\infty) = \sum_{n=1}^\infty \lambda_n f_n$  for every  $(\lambda_n)_{n=1}^\infty \in \ell^1 \subset (\ell^\infty)^*$ . So, if  $e_n$  is the usual basis element in  $\ell^1$  then  $T^*(e_n) = f_n$  so that  $(f_n)$  would have a weakly convergent subsequence. This contradicts the choice of  $(f_n)$ . Therefore,  $T$  is not weakly compact, as desired.

(a)  $\Rightarrow$  (1) In this case, every operator from  $E$  into  $F$  is weakly compact.

(b)  $\Rightarrow$  (1) By Theorem 4.2 of [4] we see that  $T$  is L-weakly compact, and by Theorem 5.61 of [1]  $T$  is well weakly compact.  $\square$

By a similar proof as the previous theorem, we obtain the following result.

**Theorem 2.6.** *Let  $X$  a Banach space and  $F$  a  $\sigma$ -Dedekind complete Banach lattice. Then the following assertions are equivalent:*

- (1) Every almost limited operator  $T : X \rightarrow F$  is weakly compact.
- (2) One of the following statements is valid:
  - (a)  $X$  is reflexive.
  - (b) The norm of  $F$  is order continuous.

As a consequence of Theorem 2.5, we obtain an operator characterization of order continuity of the norm of a  $\sigma$ -Dedekind complete Banach lattice.

**Corollary 2.7.** *Let  $E$  be a  $\sigma$ -Dedekind complete Banach lattice. Then the following statements are equivalent:*

- (1) Every almost limited operator  $T$  from  $E$  into  $E$  is weakly compact.
- (2) Every positive almost limited operator  $T$  from  $E$  into  $E$  is weakly compact.
- (3) The norm of  $E$  is order continuous.

Another consequence of Theorem 2.5 is the following result.

**Corollary 2.8.** *For a Banach lattice  $E$ , the following statements are equivalent:*

- (1) Every positive operator  $T : E \rightarrow F$  from  $E$  to an arbitrary infinite dimensional AM-space is weakly compact.
- (2) Every positive operator  $T : E \rightarrow \ell^\infty$  is weakly compact.
- (3)  $E$  is reflexive.

*Proof.* (1)  $\Rightarrow$  (2) and (3)  $\Rightarrow$  (1) are obvious.

(2)  $\Rightarrow$  (3) Follows from Theorem 2.5.  $\square$

The following result characterize Banach lattice  $E$  for which every positive almost limited operator  $T : E \rightarrow E$  has a weakly compact square.

**Theorem 2.9.** *Let  $E$  be a  $\sigma$ -Dedekind complete Banach lattice. Then the following statements are equivalent:*

- (1) Every positive almost limited operator  $T$  from  $E$  into  $E$  is weakly compact.
- (2) For every positive almost limited operator  $T$  from  $E$  into  $E$ , the operator  $T^2$  is weakly compact.
- (3) The norm of  $E$  is order continuous.

*Proof.* (1)  $\Rightarrow$  (2) Obvious.

(2)  $\Rightarrow$  (3) Assume by way of contradiction that the norm of  $E$  is not order continuous. So, by Theorem 4.14 of [1] there exists a disjoint sequence  $(u_n) \subset E^+$  satisfying  $\|u_n\| = 1$  and  $0 \leq u_n \leq u$  for all  $n$  and for some  $u \in E^+$ . We can now proceed analogously to the proof of Proposition 0.5.5 of [7]. Let  $g_n \in E_+^*$  be of norm one and such that  $g_n(u_n) = \|u_n\| = 1$  and let  $P_n$  be the band projection onto  $\{u_n\}^{dd}$ , where  $\{u_n\}^{dd}$  is the band generated by  $\{u_n\}$ . If  $f_n = g_n \circ P_n$ , then  $f_n \wedge f_m = 0$  for  $n \neq m$ ,  $\sup_n \|f_n\| \leq 1$  and  $f_n(u_m) = \delta_{nm}$ . Hence the operator  $S : \ell^\infty \rightarrow E$  defined by

$$S((t_n)_{n=1}^\infty) = (o) \sum_{n=1}^\infty t_n u_n$$

is a lattice isomorphism from  $\ell^\infty$  into  $E$ , where  $(o) \sum_{n=1}^\infty t_n u_n$  denotes the order limit of the sequence of the partial sums  $\sum_{n=1}^m t_n u_n$  for each  $(t_n)_{n=1}^\infty \in \ell^\infty$ . Also, let  $R : E \rightarrow \ell^\infty$  be the positive operator defined by

$$R(x) = (f_n(x))_{n=1}^\infty.$$

So, by Remark 2.4(2b), the positive operator  $T = S \circ R : E \rightarrow F$  defined by

$$T(x) = (o) \sum_{n=1}^{\infty} f_n(x) u_n$$

is almost limited. But  $T$  is not weakly compact. In fact, let  $x_n = \sum_{k=1}^n u_k$  for each  $n$ , and note that  $0 \leq x_n \uparrow \leq u$ . Clearly  $T(u_n) = u_n$ , and hence  $T(x_n) = x_n$  for all  $n$ . If  $x$  is a weak limit of a subsequence of  $(x_n)$ , then it is easy to see that  $x_n \uparrow x$  and  $x_n \xrightarrow{w} x$  must hold. By Theorem 3.52 of [1] we have  $\|x_n - x\| \rightarrow 0$ , and hence  $\|x_{n+1} - x_n\| \rightarrow 0$ . But this contradicts  $\|x_{n+1} - x_n\| = \|u_{n+1}\| = 1$  for all  $n$ . Thus  $(x_n)$  has no weakly convergent subsequence, and hence  $T$  is not weakly compact, as desired.

(3)  $\Rightarrow$  (1) Follows from Theorem 2.5.  $\square$

Finally, note that a weakly compact operator  $T : X \rightarrow F$  need not be almost limited. In fact, the identity operator of the Banach lattice  $\ell^2$  is weakly compact but it is not almost limited. However, if  $F$  has the positive Schur property, then the two classes coincide. The details follow.

**Proposition 2.10.** *An operator  $T : X \rightarrow F$  from a Banach space  $X$  to a Banach lattice  $F$  with the positive Schur property is weakly compact if and only if it is almost limited.*

*Proof.* The “if” part follows from Theorem 2.6. For the “only if” part, assume that  $T : X \rightarrow F$  is weakly compact. It follows from Theorem 3.4 of [3] that  $T$  is  $L$ -weakly compact, and hence  $T$  is almost limited [4, Theorem 4.2].  $\square$

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